Dynamic Power State Estimation with Asynchronous Measurements

Preprint

Guido Cavraro,[1] Emiliano Dall'Anese,[2]
and Andrey Bernstein[1]

[1] *National Renewable Energy Laboratory*
[2] *University of Colorado Boulder*

Presented at the 2019 IEEE Global Conference on Signal and Information Processing (GlobalSIP)
Ottawa, Canada
November 11–14, 2019

NREL is a national laboratory of the U.S. Department of Energy
Office of Energy Efficiency & Renewable Energy
Operated by the Alliance for Sustainable Energy, LLC

Conference Paper
NREL/CP-5D00-75064
November 2019

This report is available at no cost from the National Renewable Energy Laboratory (NREL) at www.nrel.gov/publications.

Contract No. DE-AC36-08GO28308

Dynamic Power State Estimation with Asynchronous Measurements

Preprint

Guido Cavraro,[1] Emiliano Dall'Anese,[2]
and Andrey Bernstein[1]

[1] *National Renewable Energy Laboratory*
[2] *University of Colorado Boulder*

Suggested Citation
Cavraro, Guido, Emiliano Dall'Anese, and Andrey Bernstein. 2019. *Dyanmic Power State Estimation with Asynchronous Measurements: Preprint*. Golden, CO: National Renewable Energy Laboratory. NREL/CP-5D00-75064. https://www.nrel.gov/docs/fy20osti/75064.pdf.

© 2019 IEEE. Personal use of this material is permitted. Permission from IEEE must be obtained for all other uses, in any current or future media, including reprinting/republishing this material for advertising or promotional purposes, creating new collective works, for resale or redistribution to servers or lists, or reuse of any copyrighted component of this work in other works.

NREL is a national laboratory of the U.S. Department of Energy
Office of Energy Efficiency & Renewable Energy
Operated by the Alliance for Sustainable Energy, LLC

This report is available at no cost from the National Renewable Energy Laboratory (NREL) at www.nrel.gov/publications.

Contract No. DE-AC36-08GO28308

Conference Paper
NREL/CP-5D00-75064
November 2019

National Renewable Energy Laboratory
15013 Denver West Parkway
Golden, CO 80401
303-275-3000 • www.nrel.gov

NOTICE

This work was authored in part by the National Renewable Energy Laboratory, operated by Alliance for Sustainable Energy, LLC, for the U.S. Department of Energy (DOE) under Contract No. DE-AC36-08GO28308 supported in part by the Laboratory Directed Research and Development (LDRD) Program at NREL. The views expressed herein do not necessarily represent the views of the DOE or the U.S. Government. The U.S. Government retains and the publisher, by accepting the article for publication, acknowledges that the U.S. Government retains a nonexclusive, paid-up, irrevocable, worldwide license to publish or reproduce the published form of this work, or allow others to do so, for U.S. Government purposes.

This report is available at no cost from the National Renewable Energy Laboratory (NREL) at www.nrel.gov/publications.

U.S. Department of Energy (DOE) reports produced after 1991 and a growing number of pre-1991 documents are available free via www.OSTI.gov.

Cover Photos by Dennis Schroeder: (clockwise, left to right) NREL 51934, NREL 45897, NREL 42160, NREL 45891, NREL 48097, NREL 46526.

NREL prints on paper that contains recycled content.

Dynamic Power Network State Estimation with Asynchronous Measurements

Guido Cavraro
National Renewable Energy Laboratory
Golden, CO, USA
guido.cavraro@nrel.gov

Emiliano Dall'Anese
University of Colorado Boulder
Boulder, CO, USA
emiliano.dallanese@colorado.edu

Andrey Bernstein
National Renewable Energy Laboratory
Golden, CO, USA
andrey.bernstein@nrel.gov

Abstract—**The operation of distribution networks is becoming increasingly volatile, due to fast variations of renewables and, hence, net-loading conditions. To perform a reliable state estimation under these conditions, this paper considers the case where measurements from meters, phasor measurement units, and distributed energy resources are collected and processed in real time to produce estimates of the state at a fast time scale. Streams of measurements collected in real time and at heterogenous rates render the underlying processing asynchronous, and poses severe strains on workhorse state estimation algorithms. In this work, a real-time state estimation algorithm is proposed, where data are processed on the fly. Starting from a regularized least-squares model, and leveraging appropriate linear models, the proposed scheme boils down to a linear dynamical system where the state is updated based on the previous estimate and on the measurement gathered from a few available sensors. The estimation error is shown to be always bounded under mild condition. Numerical simulations are provided to corroborate the analytical findings.**

I. Introduction

The integration of renewables, electric vehicles, and other power-electronics-interfaced distributed energy resources (DERs) are leading to net-loading conditions in distribution network that are less predictable and highly variable [1]. In these conditions, recent efforts are looking at revisiting Distribution System State Estimation (DSSE) – a fundamental task for distribution systems operators (DSOs) – to provide estimates of the state at faster time scales. Examples of DSSE include the Bayesian linear state estimator [2] and approaches based on Kalman filtering (suitable for the when a dynamical model of the network is available) [3], [4]. Current industrial and utility practices rely on approaches that produce state estimates at the minute scale (or even every 15 minutes). However, measurements from meters, phasor measurement units (PMUs) [5], [6], and distributed energy resources (DERs) could in principle be processed in real time to produce estimates at a faster time scale. Towards this end, a challenge is related to the fact that measurements provided by these devices are generally not synchronized and the difference between measurement times can be significant [7], [8]; moreover, due to the limited available bandwidth, sensors do not transmit data to the DSO all at the same time; rather, data are gathered *asynchronously*. As a consequence, the number of measurements is smaller than the number of state variables at every time step, and hence traditional state estimation algorithms cannot be straightforwardly applied. One possible strategy is to treat synchronization errors as an additional source of uncertainty and errors [9]. In [8], the measurement variances are adjusted based on the idea that measurements become less reliable as time passes, presuming that load variations can be modeled as Gaussian random variables. A robust Kalman Filter approach able to deal with bad data has been proposed in [4], [10].

To handle asynchronous measurements effectively, this paper proposes a real-time algorithm that, by leveraging a linearized distribution grid model, updates the distribution grid state estimate iteratively processing measurements as they come in. In the proposed strategy, the new estimate is computed as the solution of a strictly convex optimization problem, aiming at minimizing the sum of a weighted least square term capturing the new data and a regularizer that introduces "memory" on the estimate; this momentum term is critical for our scheme to ensure a consistent estimate even in the presence of asynchronous measurements. It turns out that the state estimate follows a standard dynamical linear system, having as an input the measurements gathered by the DSO. The estimation error is shown to be in general always finite and upper bounded when the measurement noise and the grid state variability are limited. A similar approach was adopted in [11], where a *prediction-correction* method is applied to DSSE. The scheme proposed here does not require a prediction step and can handle asynchronous measurements.

Notation. lower- (upper-) case boldface letters denote vectors (matrices). Calligraphic symbols are reserved for sets. Vectors $\mathbf{e}_m$, $\mathbf{1}$, and $\mathbf{0}$ are the m-th canonical vector, the all-one vectors, and the all-zero vector of suitable dimension. Matrices $\mathbf{I}_N$ and $\mathbf{0}_N$ represent the identity matrix and the square matrix whose entries are all zero of dimension N. Given a vector $\mathbf{x}$, $\|\mathbf{x}\|$ denotes the ℓ_2-norm and $\mathrm{dg}(\mathbf{x})$ the matrix whose diagonal is $\mathbf{x}$; given a symmetric positive-definite matrix $\mathbf{R}$, $\|\mathbf{x}\|^2_{\mathbf{R}^{-1}}$ is the weighted squared norm $\mathbf{x}^\top\mathbf{R}^{-1}\mathbf{x}$. Given a set of matrices

This work was authored in part by the National Renewable Energy Laboratory (NREL), operated by Alliance for Sustainable Energy, LLC, for the U.S. Department of Energy (DOE) under Contract DE-AC36-08GO28308 and supported in part by the Laboratory Directed Research and Development (LDRD) Program at NREL. The views expressed in the paper do not necessarily represent the views of the DOE or the U.S. Government. The U.S. Government retains and the publisher, by accepting the paper for publication, acknowledges that the U.S. Government retains a nonexclusive, paid-up, irrevocable, worldwide license to publish or reproduce the published form of this work, or allow others to do so, for U.S. Government purposes.

This report is available at no cost from the National Renewable Energy Laboratory (NREL) at www.nrel.gov/publications.

$\{\mathbf{X}_i\}_{i=1}^T$, $\mathrm{bdg}(\{\mathbf{X}_i\}_{i=1}^T)$ is the block diagonal matrix having the $\mathbf{X}_i$'s as blocks in the diagonal. The eigenvalues of a matrix $\mathbf{X}$ are collected in the set $\mathrm{eig}(\mathbf{X})$.

II. Distribution network model

A radial power distribution grid having $N+1$ buses can be modeled by a graph $\mathcal{G}_o = (\mathcal{N}, \mathcal{L})$. Nodes in $\mathcal{N} := \{0, \ldots, N\}$ represent grid buses, and the edges in $\mathcal{L}$ correspond to the L distribution lines. The active and the reactive power injection at bus n are denoted by p_n and q_n, while its voltage magnitude and its voltage phase as v_n and θ_n. The substation bus is indexed by $n = 0$ and it is modeled as a *slack bus* whose voltage is fixed at $v_0 = 1$ and $\theta_0 = 0$. Every other bus n is modeled as a constant power or P-Q bus. Powers corresponding to loads (generators) are such that $p_n \leq 0$ ($p_n \geq 0$). The voltage magnitudes, voltage angles and power injections at all buses excluding the substation are collected in the vectors $\mathbf{v}, \boldsymbol{\theta}, \mathbf{p}, \mathbf{q} \in \mathbb{R}^N$. Let $r_\ell + ix_\ell$ be the impedance of line ℓ, and collect all the impedances in vector $\mathbf{r} + i\mathbf{x}$. The grid connectivity is captured by the branch-bus incidence matrix $\tilde{\mathbf{A}} \in \{0, \pm 1\}^{L \times (N+1)}$ that can be partitioned into its first and the rest of its columns as $\tilde{\mathbf{A}} = [\mathbf{a}_0 \ \mathbf{A}]$. The *reduced bus admittance matrix* $\mathbf{Y} \in \mathbb{C}^{N \times N}$ is defined as $\mathbf{Y} := \mathbf{A}^\top \mathrm{dg}(\mathbf{r} + i\mathbf{x})^{-1}\mathbf{A}$; $\mathbf{Y}$ is symmetric, positive semi-definite and, if the network is connected, invertible. Power injections are non-linearly related to nodal voltage phasors; however, after linearizing complex power injections around the flat voltage profile $\mathbf{1} + j\mathbf{0}$, the bus voltage deviations $\tilde{\mathbf{v}} := \mathbf{v} - \mathbf{1}$ and the bus voltage angles can be approximated by [12]

$$\begin{bmatrix} \tilde{\mathbf{v}} \\ \boldsymbol{\theta} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{X} \\ \mathbf{X} & -\mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} \tag{1}$$

where $\mathbf{R} := \big(\mathrm{real}(\mathbf{Y})\big)^{-1}, \mathbf{X} := \big(\mathrm{imag}(\mathbf{Y})\big)^{-1}$. Trivially, from (1) it follows that

$$\begin{bmatrix} \tilde{\mathbf{v}} \\ \boldsymbol{\theta} \\ \mathbf{p} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{X} \\ \mathbf{X} & -\mathbf{R} \\ \mathbf{I}_N & \mathbf{0}_N \\ \mathbf{0}_N & \mathbf{I}_N \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} = \boldsymbol{\Phi} \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix}. \tag{2}$$

where we introduced the matrix $\boldsymbol{\Phi} \in \mathbb{R}^{4N \times 2N}$.

In our work, we assume that two kinds of metering devices are used: conventional smart meters, able to measure power injections and voltage magnitudes, and PMUs, able to measure power injections and both voltage magnitudes and angles. Buses endowed with smart meters are collected in the set $\mathcal{M}_{\mathrm{SM}}$, while buses endowed with PMUs in the set $\mathcal{M}_{\mathrm{PMU}}$. Let $\{t_k\}_{k \geq 1}$ be the times at which the DSO receives field data. Precisely, at time t_k, the DSO gathers measurements from a subset of buses, collected in the set $\mathcal{S}(k)$. Without loss of generality, we assume that, at each time t_k, measurements from S buses are retrieved and stacked in the vector $\mathbf{y}(k)$, i.e., $\mathcal{S}(k) = \{s_1(k), \ldots, s_S(k)\}$. Then,

$$\mathbf{y}(k) = \mathbf{S}(k)[\tilde{\mathbf{v}}^\top(k) \ \ \boldsymbol{\theta}^\top(k) \ \ \mathbf{p}^\top(k) \ \ \mathbf{q}^\top(k)]^\top + \mathbf{n}(k) \tag{3}$$

where $\mathbf{n}(k)$ represents measurement noise and where $\mathbf{S}(k)$ is a selection matrix that picks from the vector $\begin{bmatrix}\tilde{\mathbf{v}}^\top(k) & \boldsymbol{\theta}^\top(k) & \mathbf{p}^\top(k) & \mathbf{q}^\top(k)\end{bmatrix}^\top$ the quantities measured at time t_k. Matrix $\mathbf{S}(k)$ can be written as

$$\mathbf{S}(k) = \begin{bmatrix}\mathbf{S}_{s_1}^\top & \ldots & \mathbf{S}_{s_S}^\top\end{bmatrix}^\top$$

where every $\mathbf{S}_{s_n}$ can be defined in two ways:

- if $s_n \in \mathcal{M}_{SM}$, then $\mathbf{S}_{s_n} \in \{0,1\}^{3 \times 4N}$

$$\mathbf{S}_{s_n} = \begin{bmatrix} \mathbf{e}_{s_n}^\top & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top \\ \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{e}_{s_n}^\top & \mathbf{0}^\top \\ \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{e}_{s_n}^\top \end{bmatrix} \tag{4}$$

- if $s_n \in \mathcal{M}_{PMU}$, then $\mathbf{S}_{s_n} \in \{0,1\}^{4 \times 4N}$

$$\mathbf{S}_{s_n} = \begin{bmatrix} \mathbf{e}_{s_n}^\top & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top \\ \mathbf{0}^\top & \mathbf{e}_{s_n}^\top & \mathbf{0}^\top & \mathbf{0}^\top \\ \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{e}_{s_n}^\top & \mathbf{0}^\top \\ \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{e}_{s_n}^\top \end{bmatrix} \tag{5}$$

Denote the size of $\mathbf{y}(k)$, i.e., the number of measurements that arrive at every time, as M_k. The value of M_k varies as a function of the type of reporting metering devices. For instance, if at time t_k the DSO gathers measurements from C buses in $\mathcal{M}_{\mathrm{SM}}$ and from $S - C$ buses in $\mathcal{M}_{\mathrm{PMU}}$, then $M_k = 3C + 4(S - C)$. In a synchronous setting, $S(k)$ would be constant over time. Finally, the measurement noise $\mathbf{n}(k)$ is assumed to be zero-mean with diagonal covariance $\mathbf{N}_k$.

III. A DSSE Algorithm

In this section, we devise an algorithm aiming at estimate the state of the grid exploiting the measurements gathered from S buses. Nodal power injections constitute the state of the grid, described by the vector $\mathbf{x} \in \mathbb{R}^{2N}, \mathbf{x} := [\mathbf{p}^\top \ \ \mathbf{q}^\top]^\top$. Let the state of the network at time t_k be denoted as $\mathbf{x}(k)$. By combining (2) with (3), we obtain the following linear measurement model

$$\mathbf{y}(k) = \mathbf{S}(k)\boldsymbol{\Phi}\mathbf{x}(k) + \mathbf{n}(k). \tag{6}$$

Recall that measurements are processed as they come in, and that $\mathbf{y}(k)$ carries information of a limited number of buses. We make the following assumption on measurements acquisition.

Assumption 1. *There exists a constant $\tau > 0$ such that the DSO gathers measurements from every bus n at least once in the interval $[t_k, t_{k+1}, \ldots t_{k+\tau}]$, for every $k = 1, 2, \ldots$.*

Denote by $\hat{\mathbf{x}}(k)$ the estimate of the grid state at time t_k. The DSO update the state estimate after the new set of measurement $\mathbf{y}(k)$ arrives. Precisely, the new estimate is chosen as the solution of the optimization problem

$$\hat{\mathbf{x}}(k) = \arg\min_{\mathbf{w}} \|\mathbf{y}(k) - \mathbf{S}(k)\boldsymbol{\Phi}\mathbf{w}\|^2_{\mathbf{N}_k^{-1}} + \gamma\|\mathbf{w} - \hat{\mathbf{x}}(k-1)\|^2 \tag{7}$$

where $\gamma > 0$ is the *inertia parameter*. Note that

- the first term of the cost in (7) is a classical weighted linear least square term. However, it is not necessarily strictly convex, e.g., consider the targeted case when the number of measurements M_k is smaller than the state size $2N$. Hence, if $\gamma = 0$, i.e., if we neglect the second term, problem (7) may have infinite solutions.

This report is available at no cost from the National Renewable Energy Laboratory (NREL) at www.nrel.gov/publications.

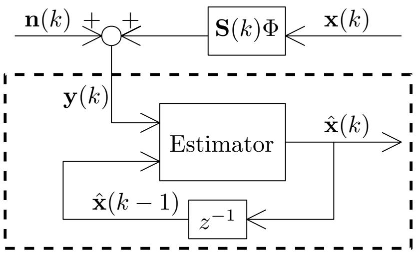

Fig. 1. Block scheme of the dynamical system described by equation (6) and equation (8).

- The second term of the cost in (7) acts as a regularizer which penalizes the Euclidean distance of the new estimate from the older one. The regularizer makes (7) a strictly convex problem and hence $\hat{\mathbf{x}}(k)$ well defined.
- The smaller the inertia parameter γ is, the further the new estimate $\hat{\mathbf{x}}(k)$ is allowed to be from $\hat{\mathbf{x}}(k-1)$.
- Task (7) is an unconstrained optimization problem. In some cases, prior information can be leveraged to restrict the state space, e.g., by adding constraints, and improve the estimation accuracy. This is left for our future works.

Note that the new estimate can be explicitly written as

$$\hat{\mathbf{x}}(k) = \mathbf{\Lambda}(k)\hat{\mathbf{x}}(k-1) + \frac{1}{\gamma}\mathbf{\Lambda}(k)\mathbf{\Phi}^\top\mathbf{S}(k)^\top\mathbf{N}_k^{-1}\mathbf{y}(k) \quad (8)$$

where

$$\mathbf{\Lambda}(k) = \gamma(\mathbf{\Phi}^\top\mathbf{S}(k)^\top\mathbf{N}_k^{-1}\mathbf{S}(k)\mathbf{\Phi} + \gamma\mathbf{I})^{-1}. \quad (9)$$

Matrix $\mathbf{\Lambda}(k)$ is always symmetric and positive definite. Equations (6) and (8) represent a linear dynamical system, whose block scheme is reported in Figure 1. Furthermore, heed that equation (8) is essentially a classic closed-loop system, see the dashed area in Figure 1. Such system features the ensuing stability property, proved in the Appendix.

Proposition 1. *Let Assumption 1 hold. Define the state variation* $\mathbf{\Delta}(k) = \mathbf{x}(k) - \mathbf{x}(k-1)$*, the estimation error* $\boldsymbol{\xi}(k) = \hat{\mathbf{x}}(k) - \mathbf{x}(k)$*, and the scalar values* $\sigma = \max_k\{\lambda \in \mathrm{eig}(\mathbf{\Lambda}(k)), \lambda \neq 1\}$*. Then,*

1) the system (8) *is asymptotically stable. In particular,* $\sigma < 1$ *and, for* $k \geq 1$*,*

$$\|\hat{\mathbf{x}}(k+\tau)\| \leq \sigma\|\hat{\mathbf{x}}(k)\| \quad (10)$$

2) the system (8) *is bounded input - bounded output (BIBO) stable*

3) if the state variation norm and the measurement noise norm are upper-bounded, i.e., $\|\mathbf{\Delta}(k)\| \leq \delta_x, \|\mathbf{n}(k)\| \leq \delta_n, \forall k$*, the estimation error satisfies*

$$\lim_{k\to\infty}\|\boldsymbol{\xi}(k)\| \leq \tau\Big(\delta_x + \frac{c}{\gamma}\delta_n\Big)\Big(1 + \frac{\gamma}{\sigma}\Big). \quad (11)$$

Proposition 1-2) implies that if the sequence of measurements $\{\mathbf{y}(k)\}_k$ is bounded, then the sequence of estimates $\{\hat{\mathbf{x}}(k)\}_k$ does not diverge. On the other hand, Proposition 1-3) upper bounds the estimation error when state variation and measurement norm are bounded. Note that the latter scenario includes the case in which $\mathbf{n}$ is stochastic with limited support, e.g., $\mathbf{n}$ is a random vector drawn from a uniform distribution or drawn from a truncated Gaussian distribution.

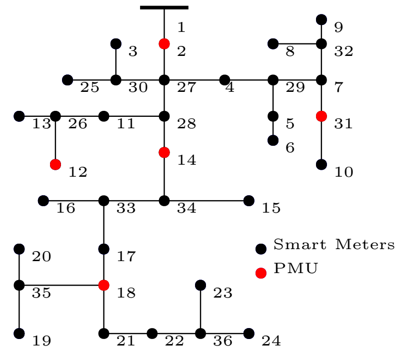

Fig. 2. Schematic representation of the IEEE 37-bus test feeder.

IV. Numerical Tests

Our estimation algorithm was validated on the IEEE 37-bus feeder converted to its single-phase equivalent, depicted in Figure 2. Measurements are taken from the devices and gathered from the system operator once every 10 seconds. Loads were generated by adding a zero-mean Gaussian variation to the benchmark data, with standard deviation 0.22 times the nominal loads [13]. Voltages were obtained via a power flow solver and then corrupted by a truncated zero-mean Gaussian noise with 3σ deviation of 1% per unit (pu) [14]. Every bus in the network is endowed either with a smart meter or with a PMU, see their location in Figure 2. The algorithm was tested for different values of the inertia parameter γ and for different numbers of reporting metering devices S. Each scenario has been studied through 200 Monte Carlo simulations. The state estimate was always initialized at $\hat{\mathbf{x}}(0) = \mathbf{0}$. The S reporting devices were randomly chosen at each algorithm iteration. However, every device was forced to report data at least once every 100 iterations. Define the average relative estimation error $e(k)$ as the average computed over the Monte Carlo simulations of the relative estimation error, i.e., $e(k) = \log\Big(\mathbb{E}\Big[\|\boldsymbol{\xi}(k)\|/\|\mathbf{x}(k)\|\Big]\Big)$.

Figure 3 reports $e(k)$ for different values of the inertia parameter γ, when $S = 4$. In general, the smaller is γ, the faster the algorithm error reaches its asymptotic value. For what concerns the asymptotic error, the best performance are obtained when $\gamma = 1$. This can be understood by looking at Figure 4, which compares the active power absorbed by bus 23 (denoted as p_{23}) with its estimates (denoted by $\hat{x}_{23}$) in one of the Monte Carlo runs for different values of γ. When $\gamma > 1$, the regularizer term in (6) is dominant. Hence, $\hat{x}_{23}(k)$ is forced to be close to the old estimate $\hat{x}_{23}(k-1)$ and is slowly chasing the state p_{23}. On the other hand, when $\gamma < 1$, Figure 4 shows high fluctuations of $\hat{x}_{23}(k)$. In fact, the estimate becomes more sensitive to measurement noise.

Finally, Figure 5 shows how the estimator performs for different numbers of reporting meters S, when $\gamma = 1$. Not surprisingly, the bigger S is, the better is the performance.

This report is available at no cost from the National Renewable Energy Laboratory (NREL) at www.nrel.gov/publications.

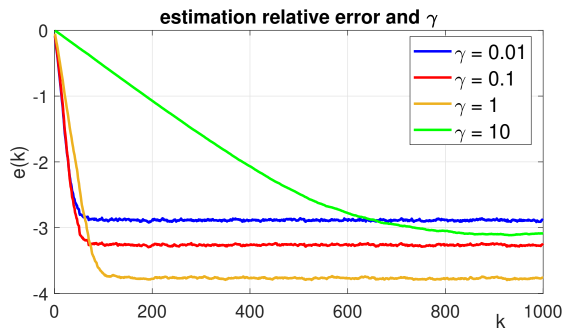

Fig. 3. Average estimation error for different values of γ, with $S = 4$.

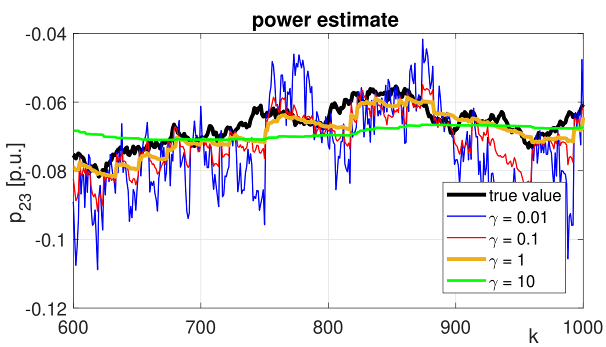

Fig. 4. Estimation error for a particular algorithm run, with $S = 4$.

V. Conclusion

We have proposed a state estimation algorithm for power distribution grids able to deal with asynchronous measurements. Simulations on the standard IEEE-37 bus testbed have shown the effectiveness of the proposed strategy. Generalizations to non-linear measurements-state relation and considering a constrained optimization problem rather than (6) constitute interesting research directions.

Appendix

The proof of Proposition 1 will use the next Lemma and, due to space constraints, it is only sketched.

Lemma 1. *Let Assumption 1 holds. Then,*

$$\bigcap_{j=0}^{\tau} \ker\Big(\mathbf{S}(k+j)\boldsymbol{\Phi}\Big) = \mathbf{0}. \tag{12}$$

Proof of Proposition 1: Firstly, heed that, being $\boldsymbol{\Phi}^\top \mathbf{S}(k)^\top \mathbf{N}_k^{-1}\mathbf{S}(k)\boldsymbol{\Phi}$ a symmetric positive-semidefinite matrix, it can be written as

$$\boldsymbol{\Phi}^\top \mathbf{S}(k)^\top \mathbf{N}_k^{-1}\mathbf{S}(k)\boldsymbol{\Phi} = [\mathbf{U}(k)\ \mathbf{V}(k)]\,\mathrm{bdg}(\boldsymbol{\Sigma}(k), \mathbf{0}_{N-M})[\mathbf{U}(k)\ \mathbf{V}(k)]^\top$$

where $\boldsymbol{\Sigma}(k)$ is the diagonal matrix collecting the eigenvalues of $\boldsymbol{\Phi}^\top \mathbf{S}(k)^\top \mathbf{N}_k^{-1}\mathbf{S}(k)\boldsymbol{\Phi}$. Columns of $\mathbf{V}(k)$ are eigenvectors spanning $\ker(\boldsymbol{\Phi}^\top \mathbf{S}(k)^\top \mathbf{N}_k^{-1}\mathbf{S}(k)\boldsymbol{\Phi}) = \ker(\mathbf{S}(k)\boldsymbol{\Phi})$. Then, it follows that

$$\boldsymbol{\Lambda}(k) = [\mathbf{U}(k)\ \mathbf{V}(k)]\,\mathrm{bdg}(\tilde{\boldsymbol{\Sigma}}(k), \mathbf{I}_{N-M})[\mathbf{U}(k)\ \mathbf{V}(k)]^\top. \tag{13}$$

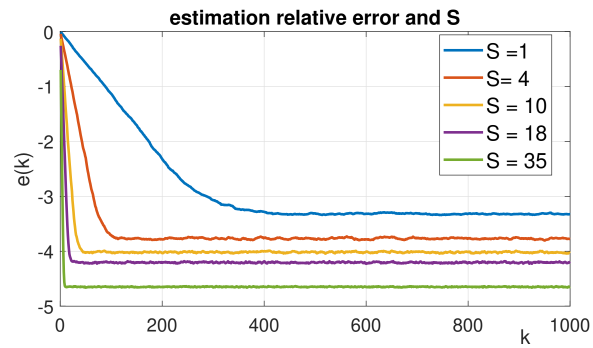

Fig. 5. Average estimation error for different values of S, with $\gamma = 1$.

where

$$\tilde{\boldsymbol{\Sigma}}(k) = \gamma(\gamma\mathbf{I} + \boldsymbol{\Sigma}(k))^{-1}. \tag{14}$$

Note that $\tilde{\boldsymbol{\Sigma}}(k)$ is a diagonal matrix whose entries are positive and stricly lower than 1, and $\sigma = \max_k \mathrm{eig}(\tilde{\boldsymbol{\Sigma}}(k))$. Matrix $\mathbf{V}(k)$ collects all the eigenvectors of $\boldsymbol{\Lambda}(k)$ associated with the eigenvealue 1. Given any $\hat{\mathbf{x}}(k)$

$$\hat{\mathbf{x}}(k+\tau) = \prod_{j=k}^{k+\tau} \boldsymbol{\Lambda}(j)\hat{\mathbf{x}}(k)$$
$$= \prod_{j=k}^{k+\tau} [\mathbf{U}(j)\ \mathbf{V}(j)]\,\mathrm{bdg}(\tilde{\boldsymbol{\Sigma}}(j), \mathbf{I}_{N-M})[\mathbf{U}(j)\ \mathbf{V}(j)]^\top \hat{\mathbf{x}}(k). \tag{15}$$

Since Assumption 1 and Lemma 1 implies that

$$\prod_{j=k}^{k+\tau} \mathbf{V}(j)\mathbf{V}(j)^\top = \mathbf{0}_N, \tag{16}$$

by combining equations (15) and (16), and by exploiting the properties of norms, we obtain

$$\Big\| \prod_{j=k}^{k+\tau} \boldsymbol{\Lambda}(j) \Big\| \le \sigma \tag{17}$$

from which equation (10) follows.

Concerning the BIBO stability, note that, iterating equation (8) yields

$$\hat{\mathbf{x}}(T) = \prod_{k=1}^{T} \boldsymbol{\Lambda}(k)\hat{\mathbf{x}}(0) + \gamma^{-1}\sum_{k=1}^{T}\prod_{j=k}^{T} \boldsymbol{\Lambda}(j)\boldsymbol{\Phi}^\top \mathbf{S}(j)^\top \mathbf{N}_j^{-1}\mathbf{y}(j) \tag{18}$$

Let $y_{\max} = \max_k\{\gamma^{-1}\|\mathbf{y}(k)\|\}$. The triangle inequality applied to (18) yields

$$\|\hat{\mathbf{x}}(T)\| \le \Big\|\prod_{k=1}^{T}\boldsymbol{\Lambda}(k)\Big\|\|\hat{\mathbf{x}}(0)\| + y_{\max}\sum_{k=1}^{T}\Big\|\prod_{j=k}^{T}\boldsymbol{\Lambda}(j)\Big\| \tag{19}$$
$$\le \|\hat{\mathbf{x}}(0)\| + y_{\max}\rho\tau\sum_{i=0}^{\mu}\sigma^i$$
$$\le \|\hat{\mathbf{x}}(0)\| + y_{\max}\rho\tau\frac{1}{1-\sigma}$$

where $\mu = \mod (T/\tau)$ and $\rho = \max_k \|\boldsymbol{\Phi}^\top \mathbf{S}(k)^\top \mathbf{N}_k^{-1}\|$.

Finally, equation (11) can be obtained by making T in (19) tending to infinity and by using (6) and (10). ■

This report is available at no cost from the National Renewable Energy Laboratory (NREL) at www.nrel.gov/publications.

References

[1] A. Primadianto and C. Lu, "A review on distribution system state estimation," *IEEE Transactions on Power Systems*, vol. 32, no. 5, pp. 3875–3883, Sep. 2017.

[2] L. Schenato, G. Barchi, D. Macii, R. Arghandeh, K. Poolla, and A. Von Meier, "Bayesian linear state estimation using smart meters and pmus measurements in distribution grids," in *2014 IEEE International Conference on Smart Grid Communications (SmartGridComm)*, Nov 2014, pp. 572–577.

[3] C. Carquex, C. Rosenberg, and K. Bhattacharya, "State estimation in power distribution systems based on ensemble kalman filtering," *IEEE Transactions on Power Systems*, vol. 33, no. 6, pp. 6600–6610, Nov 2018.

[4] J. Zhao, M. Netto, and L. Mili, "A robust iterated extended kalman filter for power system dynamic state estimation," *IEEE Transactions on Power Systems*, vol. 32, no. 4, pp. 3205–3216, July 2017.

[5] J. Zheng, D. W. Gao, and L. Lin, "Smart meters in smart grid: An overview," in *2013 IEEE Green Technologies Conference (GreenTech)*, April 2013, pp. 57–64.

[6] A. von Meier, D. Culler, A. McEachern, and R. Arghandeh, "Micro-synchrophasors for distribution systems," Washington, DC, Feb. 2014.

[7] A. Angioni, C. Muscas, S. Sulis, F. Ponci, and A. Monti, "Impact of heterogeneous measurements in the state estimation of unbalanced distribution networks," in *2013 IEEE International Instrumentation and Measurement Technology Conference (I2MTC)*, May 2013, pp. 935–939.

[8] A. Alimardani, F. Therrien, D. Atanackovic, J. Jatskevich, and E. Vaahedi, "Distribution system state estimation based on nonsynchronized smart meters," *IEEE Transactions on Smart Grid*, vol. 6, no. 6, pp. 2919–2928, Nov 2015.

[9] S. Bolognani, R. Carli, and M. Todescato, "State estimation in power distribution networks with poorly synchronized measurements," in *53rd IEEE Conference on Decision and Control*, Dec 2014, pp. 2579–2584.

[10] M. Netto, J. Zhao, and L. Mili, "A robust extended kalman filter for power system dynamic state estimation using pmu measurements," in *2016 IEEE Power and Energy Society General Meeting (PESGM)*, July 2016, pp. 1–5.

[11] J. Song, E. Dall'Anese, A. Simonetto, and H. Zhu, "Dynamic distribution state estimation using synchrophasor data," 2019. [Online]. Available: https://arxiv.org/abs/1901.02554

[12] S. Bolognani and S. Zampieri, "On the existence and linear approximation of the power flow solution in power distribution networks," *IEEE Transactions on Power Systems*, vol. 31, no. 1, pp. 163–172, Feb. 2015.

[13] G. Cavraro and R. Arghandeh, "Power distribution network topology detection with time-series signature verification method," *IEEE Transactions on Power Systems*, vol. 33, no. 4, pp. 3500–3509, July 2018.

[14] EEI-AEIC-UTC. (2011) Smart meters and smart meter systems: A metering industry perspective. [Online]. Available: http://www.eei.org/issuesandpolicy/grid-enhancements/Documents/smartmeters.pdf

This report is available at no cost from the National Renewable Energy Laboratory (NREL) at www.nrel.gov/publications.