# Grassmannian Shape Representations for Aerodynamic Applications


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## Motivation

Airfoil shape design is a classical problem in engineering and manufacturing. Our motivation is to combine principled physics-based considerations for the shape design problem with modern computational techniques informed by a data-driven approach. Traditional analyses of airfoil shapes emphasize a flow-based sensitivity to deformations which can be represented generally by affine transformations (rotation, scaling, shearing, translation). We present a novel representation of shapes which decouples affine-style deformations from a rich set of data-driven deformations over a submanifold of the Grassmannian. This representation enhances manufacturability and design of aerodynamic shapes by providing a unified design space for 2D airfoils and enabling consistent 3D blade representations and perturbations over a sequence of nominal shapes.

## Airfoil Shape Representations & Affine Transformations

We represent a 2D shape as a boundary defined by the closed (injective) curve

$$c : \mathcal{I} \subset \mathbb{R} \to \mathbb{R}^2 : s \mapsto c(s)$$

over a compact domain $\mathcal{I}$ which can be arbitrarily reparametrized to $[0,1]$

Affine deformations of the airfoil have the form

$$\tilde{c}(s) \; = \; M^{\top} c(s) + b$$

where $M \in GL_2$ is from the set of all invertible $2 \times 2$ matrices and $b \in \mathbb{R}^2$

In practice, we represent the airfoil shape as an ordered sequence of $n$ ***landmarks***

$$(x_i) \in \mathbb{R}^2 \quad \text{for} \quad i = 1, \dots, n$$

which we can combine into a matrix

$$X = [x_1, \dots, x_n]^{\top} \in \mathbb{R}_*^{n \times 2}$$

where $\mathbb{R}_*^{n \times 2}$ refers to the space of full-rank $n \times 2$ matrices

Affine deformations of discrete shape representation can be written as the smooth right action with translation

$$\tilde{X} \; = \; XM + 1\,\mathrm{diag}(b)$$

where $1 \in \mathbb{R}^{n \times 2}$ denotes a matrix of ones

The linear term $M$ can drive four types of physically meaningful deformations as one-parameter subgroups through $GL_2$:

(i) changes in thickness

(ii) changes in camber

(iii) changes in chord

(iv) changes in twist (rotation or angle-of-attack)

## Grassmannian Shapes

The *Grassmannian* $\mathcal{G}(n,q)$ is the space of all $q$-dimensional subspaces of $\mathbb{R}^n$

→ Note that for (planar) airfoil design, we consider $q = 2$

Formally, $\mathcal{G}(n,q) \cong \mathbb{R}_*^{n \times q} / GL_q$ and $\tilde{X} \in \mathbb{R}_*^{n \times q}$ is a full-rank representative element of an equivalence class $[\tilde{X}] \in \mathcal{G}(n,q)$ of all matrices with equivalent span [1]

→ Every element of the Grassmannian is a full-rank matrix modulo $GL_q$ deformations

→ Airfoil elements of the Grassmannian are decoupled from the aerodynamically important affine deformations

*Landmark-Affine (LA) standardization* maps physical airfoil shapes as elements of the Grassmannian [2]

→ Normalizes the shape such that it has zero mean and identity covariance

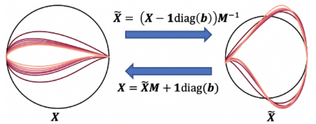
## Principal Geodesic Representations

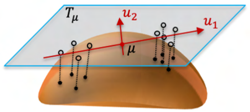
Independence of $\mathcal{G}(n,q)$ to affine deformations enables a data-driven approach to identifying high-order, physically relevant shape deformations known as ***principal geodesic analysis (PGA)*** [3]

→ Generalization of principal component analysis (PCA) over Riemannian manifolds

→ Determines principal components as elements in a ***central tangent space***, $T_{[\tilde{X}_0]}\mathcal{G}(n,2)$ of a given dataset, where $\tilde{X}_0$ is the Karcher mean over the manifold

PGA constitutes a manifold learning procedure for computing an important submanifold of $\mathcal{G}(n,2)$ representing a design space of physically relevant airfoil shapes inferred from provided data [4]

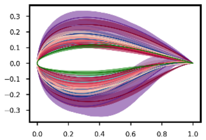



## Airfoil Example

Data considerations:

- Consider 16 baseline airfoils from the NREL 5MW, DTU 10MW, and IEA 15MW reference wind turbines [5, 6, 7]
- Identify baseline shapes' class-shape transformation (CST) representations, which encode upper and lower surfaces as a 20-term polynomial expansion [8]
- Define 1,000 perturbations of nominal CST coefficients by up to 20% for each baseline airfoil

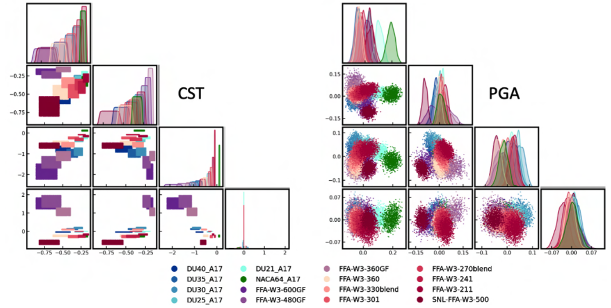


Using PGA, we reduce the full 20-dimensional CST parameter to $r = 4$ principal basis components

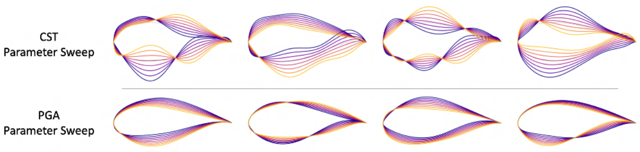
→ PGA perturbations are independent of previous affine deformations

→ Distribution of the complete design space is more unified than with CST

→ Sampling through parameters results in realistic airfoil shapes




Note: figure only shows a random selection of four CST parameters from the full 20D space for readability




## Blade Interpolation and Perturbation

Wind turbine blade designs are often characterized by an ordered set of planar airfoils at different blade-span positions from hub to tip of the blade

→ Current design approaches require significant hand-tuning of airfoils to ensure the construction of valid blade geometries without dimples or kinks

→ The Grassmannian framework enables the flexible design of new blades by applying consistent deformations to all airfoils and smooth interpolation of shapes between landmarks

We represent a discretized blade as a sequence of matrices $(X_k) \in \mathbb{R}_*^{n \times 2}$ with an induced sequence of equivalence classes over the Grassmannian

$$([\tilde{X}_k]) \in \mathcal{G}(n,2) \quad \text{for} \quad k = 1, \dots, N$$

at discrete blade-span positions $\eta_k \in \mathcal{S} \subset \mathbb{R}$

We can construct a piecewise geodesic path over the Grassmannian to interpolate discrete blade shapes independent of affine deformations [9]

$$\tilde{\gamma}_{k,k+1} : [\tilde{X}_k] \mapsto [\tilde{X}_{k+1}]$$

New blade shapes can be constructed by perturbing landmark airfoils along the span of the blade

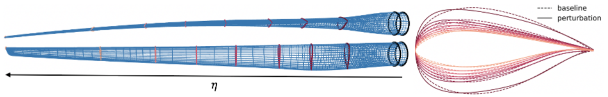
→ Typically requires careful hand-tuning of the shapes to ensure manufacturability of the blade

PGA shape perturbations are defined by a direction in the tangent space of Karcher mean, $T_{[\tilde{X}_0]}\mathcal{G}(n,2)$

We utilize an isometry called ***parallel transport*** to smoothly ``translate" the perturbing vector field along separate geodesics connecting the Karcher mean to each of the individual ordered airfoils

$$T_{[\tilde{X}_k]}\mathcal{G}(n,2)$$

This results in a natural framework for interpolating 2D shapes into 3D blades and the decoupling of affine and higher-order deformations make Grassmann-based shape representation a powerful tool enabling AI/ML-driven aerodynamic design

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