

Generalized filtered lifting line theory for arbitrary chord lengths and application to wind turbine blades

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Funding information

National Science Foundation, Grant/Award Numbers: 1949778, 2034111; US Department of Energy (DOE), Grant/Award Numbers: DE-AC36-08GO28308, DE-NA0003525

Abstract

The filtered lifting line theory is an analytical approach used to solve the equations of flow subjected to body forces with a Gaussian distribution, such as used in the actuator line model. In the original formulation, the changes in chord length along the blade were assumed to be small. This assumption can lead to errors in the induced velocities predicted by the theory compared to full solutions of the equations. In this work, we revisit the original derivation and provide a more general formulation that can account for significant changes in chord along the blade. The revised formulation can be applied to wings with significant changes in chord along the span, such as wind turbine blades.

KEYWORDS

aerodynamics, actuator line model, lifting line

1 | INTRODUCTION

The filtered lifting line theory was developed to obtain solutions of flow subjected to Gaussian body forces, as used in the actuator line model formulation.^{1,2} The original formulation assumed that the changes in chord along the blade were small, making it possible to express the induced velocity analytically as an integral convolution equation. However, this assumption does not hold for many wings, specifically wind turbine blades. In this note, we propose a generalization of the original formulation to take into account strong changes in chord (and Gaussian smoothing kernel ϵ) along the blade. The generalized theory is also analytical and preserves the usefulness of the filtered lifting line theory and its applicability to the actuator line model. Also, we apply the approach by simulating a fixed wing with a chord distribution similar to a wind turbine blade. We compare the results using the filtered lifting line theory correction to solutions from large-eddy simulations (LESs) using an actuator line. In this work, we highlight the differences between the corrected and uncorrected cases, implement the new formulation, rerun simulations, and highlight the improvements when using the generalized formulation.

We focus on the fixed wing case illustrated in Figure 1. The inflow is in the x direction, y is normal to the inflow, and z is the coordinate along the blade. The induced velocities from the filtered lifting line theory are in the y direction.

2 | FILTERED LIFTING LINE THEORY CORRECTION

The filtered lifting line theory expresses the induced velocity along a wing, $u'_y(z)$, as the solution to the integro-differential equation¹:

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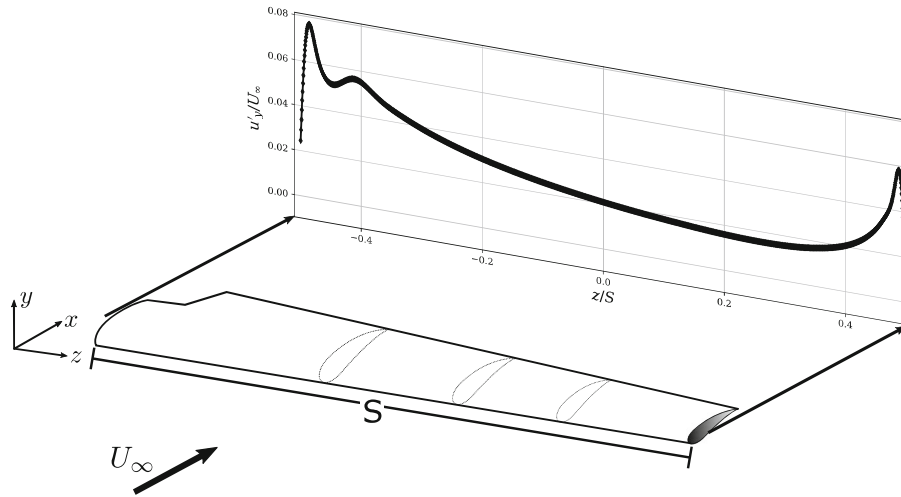


FIGURE 1 Diagram of flow over a wing with the induced velocity from an actuator line simulation plotted downstream.

$$u'_y(z) = - \int_0^S \frac{dG(z')}{dz'} \frac{(1 - e^{-(z-z')^2/\epsilon(z')^2})}{U_\infty(z') 4\pi(z-z')} dz', \quad (1)$$

where $G(z') = \frac{1}{2} c_L(\alpha) c W^2$ is the lift force per unit span, U_∞ is the inflow velocity, $W = \sqrt{U_\infty^2 + u_y'^2}$ is the magnitude of the velocity vector, S is the wing span, c_L is the lift coefficient, α is the angle of attack that depends on $u_y'(z')$, c is the chord, ϵ is the width of the body force kernel and/or vortex core size, and z is the spanwise coordinate.

The width of the forces distributed in the filtered lifting line formulations is given by ϵ . Previous work suggests that a value of ϵ on the order of $0.25c$ provides velocity distributions arising from the Gaussian forcing that most closely resemble those of ideal flow over a lifting airfoil with chord length, c .^{1,3} Equation (1) is then used to correct coarse-grained simulations, typically LES, that cannot afford the grid resolutions required to run with the optimal width ($\epsilon_{\text{opt}} = 0.25c$). The correction is done by computing the difference between the induced velocity in the LES (from using ϵ_{LES}) and the desired optimal (ϵ_{opt}) and adding the difference to the velocity sampled from the LES. This approach has recently been used to reduce the dependency on grid resolution and ϵ from simulations of fixed wings and wind turbine blades using the actuator line model.^{1,4-8}

3 | SHORTCOMINGS OF THE THEORY

Here we focus on a case in which the correction has significant shortcomings. We performed LES of a wing that has a chord distribution similar to a wind turbine blade using the actuator line model. We use AMR-Wind, an incompressible LES code meant for wind energy applications using structured grids and adaptive mesh refinement.^{9,10} We performed simulations of a fixed wing in uniform inflow with a baseline angle of attack, $\alpha = 6^\circ$. We used a large domain ($16S$ in all directions) with grid refinements to ensure that the effect on induced velocities from domain size is less than 0.1%. The grid spacing (and levels of refinement) is set to always meet the criteria of $\epsilon/\Delta > 4$.¹¹ Our tests indicated that there is no difference in the induced velocities by further increasing the resolution. The simulations use 400 actuator points along the blade, and $\epsilon/S = 0.1, 0.2, 0.4, 0.6$ (typical values used in wind turbine simulations using the actuator line model).

Figure 2 shows the induced velocity from LES with and without the filtered lifting line theory correction compared to an uncorrected simulation with fine resolution (symbols). Large differences are observed in induced velocities between the cases without the correction, as expected.¹ The cases with the correction all collapse onto each other, as expected based on the original formulation.¹ However, there is disagreement near the root of the blade, where the changes in chord occur over distances comparable to c . Careful examination helped us understand that the difference is caused by the theory's limitation assuming that the changes in chord and ϵ along the blade are small ($d\epsilon/dz < 1$). This finding motivates a new and more general formulation for the filtered lifting line theory as presented in the following section.

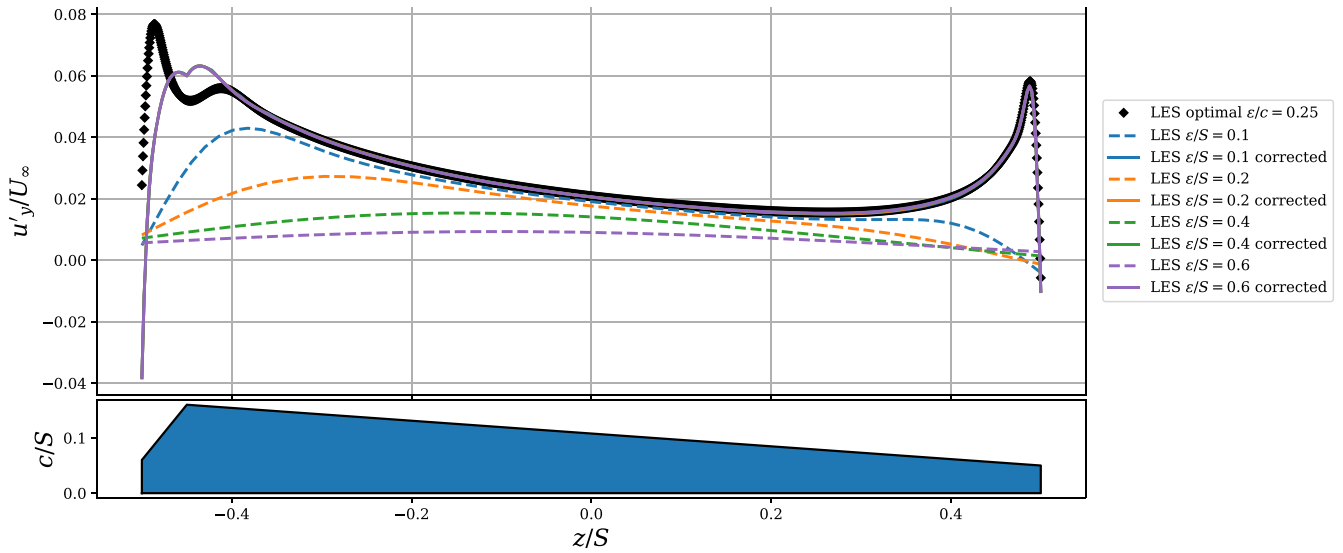


FIGURE 2 The induced velocity from the LES using the filtered lifting line theory correction does not match the results with optimal ϵ without the correction in the blade root region. The chord profile of the wing is shown at the bottom.

4 | GENERALIZED FORMULATION

We start by writing the solution to the streamwise vorticity equation from the linearized form of the vorticity transport equation¹:

$$\omega_x(x, y, z) = \frac{1}{2U_\infty(z)\pi} \frac{\partial}{\partial z} \int_0^s [\operatorname{erf}(x/\epsilon(z')) + 1] \frac{1}{\epsilon(z')^2} G(z') e^{-(y^2 + (z-z')^2)/\epsilon(z')^2} dz' \quad (2)$$

The vorticity is composed of the superposition of the curl of forces at every blade position. Quantities inside the integral (z') take into account the changes in ϵ along the span. Following Martínez-Tossas and Meneveau,¹ we use Biot-Savart to obtain the velocity component normal to the inflow along the blade span:

$$u'_y(z) = \frac{1}{8\pi^2} \iiint \frac{z-z''}{(x''^2 + y''^2 + (z-z'')^2)^{3/2}} \frac{1}{U_\infty(z'')} \frac{\partial}{\partial z} \int_0^s [\operatorname{erf}(x''/\epsilon(z')) + 1] \frac{G(z')}{\epsilon(z')^2} e^{-(y''^2 + (z''-z')^2)/\epsilon(z')^2} dz' dx'' dy'' dz'' \quad (3)$$

The original formulation in filtered lifting line theory assumes that the changes in $\epsilon(z)$ are small (i.e., $d\epsilon/dz \ll 1$) and moves them outside of the integral in z' . This assumption enabled the use of integration by parts to obtain an expression involving dG/dz (see Equation 1). To take into account the variations of ϵ along the blade, all terms should instead be kept inside the z' integration. Note that while LES may use a z -independent ϵ_{LES} , the correction also depends on $\epsilon_{\text{opt}} = 0.25c(z)$, which is a function of z if c varies along the span. Evaluating the integrals analytically (aided by the use of integral tables and Mathematica software¹²⁻¹⁴) leads to the new formulation for induced velocity in filtered lifting line theory:

$$u'_y(z) = -\frac{1}{2\pi} \int_0^s \frac{1}{U_\infty(z')} \frac{G(z')}{\epsilon(z')^2} \left[e^{-(z'-z)^2/\epsilon(z')^2} + \frac{\epsilon(z')^2}{2(z'-z)^2} \left(e^{-(z'-z)^2/\epsilon(z')^2} - 1 \right) \right] dz' \quad (4)$$

Equation (4) is the generalized form of the filtered lifting line theory equations that applies to blades with significant changes in chord along the span, such as wind turbine blades. Note that unlike Equation (1), which is an integro-differential equation, Equation (4) is an integral equation because it involves G instead of dG/dz . It is worth highlighting that in the limit of $z \rightarrow z'$, the function inside the integral tends to $G(z')/(2U_\infty(z')\epsilon(z')^2)$ (i.e., it does not diverge).

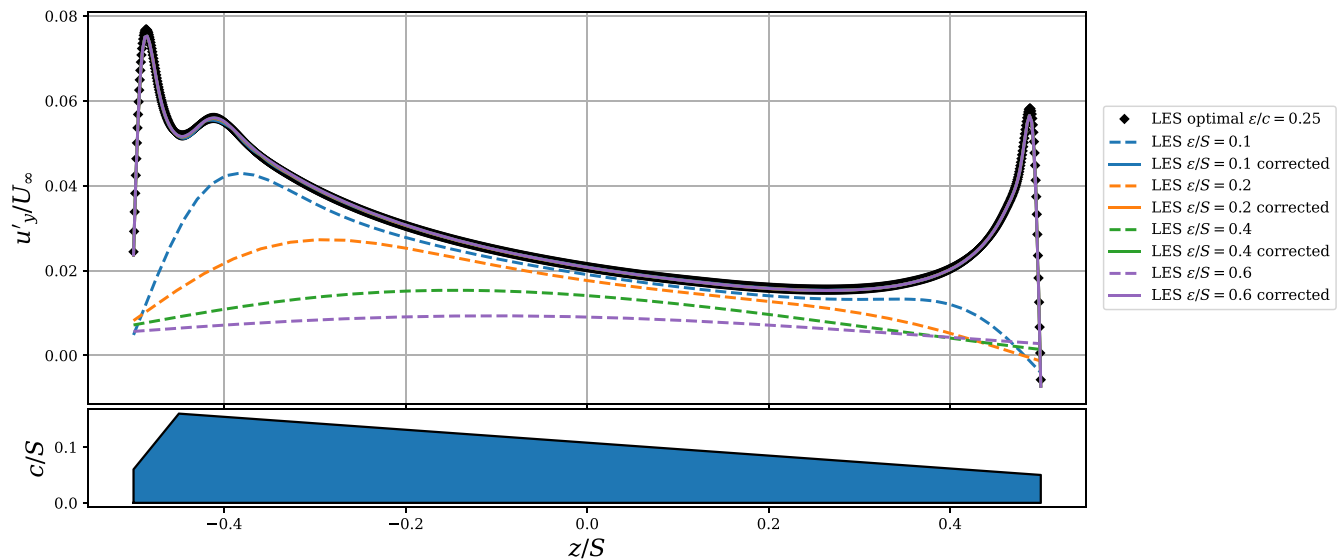


FIGURE 3 The induced velocity from the LES using the filtered lifting line theory correction with the newly developed formulation match results with optimal ϵ without the correction.

5 | IMPROVEMENTS

We used the new formulation and implemented the filtered lifting line theory correction in our LES solver using Equation (4). We performed the same simulations from Section 3 using the new formulation and compared the results to the simulation with the optimal ϵ . Figure 3 shows the induced velocity for simulations with and without the correction compared to the high-fidelity simulation without the correction. There is excellent agreement within all of the corrected simulations, with the mean difference in induced velocity being within less than 1% of the simulation using $\epsilon_{\text{opt}} = 0.25c$. These results confirm that the new generalized formulation for filtered lifting line theory can capture changes along the span with high accuracy.

6 | CONCLUSIONS

The filtered lifting line theory was developed to predict the induced velocity and forces along a wing modeled by Gaussian body forces. The original formulation made assumptions that limited the accuracy of the theory when applied to blades that have significant changes in chord along the span, such as wind turbine blades. We reformulated the filtered lifting line theory and obtained a new and more general formulation that can be applied to wings that have strong variations of chord along the blade span. We tested the new formulation using the filtered lifting line theory correction and obtained excellent agreement with high-fidelity simulations with optimal $\epsilon_{\text{opt}}(z) = 0.25c(z)$.

This note improves the original formulation of the filtered lifting line theory by relaxing its assumptions. This work represents a step toward an even more generalized filtered lifting line theory. Future work should focus on a formulation that covers swept wings, includes drag force, and can account for helical vortices.

AUTHOR CONTRIBUTIONS

Luis A. Martínez-Tossas reformulated the filtered lifting line theory equations and performed simulations. Philip Sakievich and Luis A. Martínez-Tossas implemented the software. Charles Meneveau and Matthew J. Churchfield provided feedback and helped with the derivation.

ACKNOWLEDGEMENTS

This work was authored in part by the National Renewable Energy Laboratory, operated by Alliance for Sustainable Energy, LLC, for the US Department of Energy (DOE) under Contract No. DE-AC36-08GO28308, funding provided by the US DOE Office of Energy Efficiency and Renewable Energy Wind Energy Technologies Office. The views expressed in the article do not necessarily represent the views of the DOE or the US government. The US government retains and the publisher, by accepting the article for publication, acknowledges that the US government retains a nonexclusive, paid-up, irrevocable, worldwide license to publish or reproduce the published form of this work, or allow others to do so, for US government purposes. Sandia National Laboratories is a multimission laboratory managed and operated by National Technology &

Engineering Solutions of Sandia, LLC, a wholly owned subsidiary of Honeywell International Inc., for the US DOE's National Nuclear Security Administration under contract DE-NA0003525. (A portion of) The research was performed using computational resources sponsored by the US DOE's Office of Energy Efficiency and Renewable Energy and located at the National Renewable Energy Laboratory. C.M. acknowledges support from the National Science Foundation (1949778 and 2034111).

CONFLICT OF INTEREST STATEMENT

The authors declare no potential conflict of interests.

PEER REVIEW

The peer review history for this article is available at <https://www.webofscience.com/api/gateway/wos/peer-review/10.1002/we.2872>.

DATA AVAILABILITY STATEMENT

Data are available on request from the authors.

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How to cite this article: Martínez-Tossas LA, Sakievich P, Churchfield MJ, Meneveau C. Generalized filtered lifting line theory for arbitrary chord lengths and application to wind turbine blades. *Wind Energy.* 2024;27(1):101-106. doi:10.1002/we.2872

APPENDIX A: ORIGINAL LIFTING LINE THEORY

In the limit of $\epsilon \rightarrow 0$, the filtered lifting line theory equations are reduced to

$$u'_y(z) = \frac{1}{4\pi} \int_0^s \frac{1}{U_\infty(z')} \frac{G(z')}{(z' - z)^2} dz'$$

Using integration by parts, the integral reduces to the same formulas in the original lifting line theory, making both theories consistent in the limit of infinitesimal vorticity.

$$u'_y(z) = \frac{1}{4\pi} \int_0^s \frac{\partial G(z')}{\partial z'} \frac{1}{U_\infty(z')(z' - z)} dz'.$$

APPENDIX B: FILTERED LIFTING LINE THEORY CORRECTION

The induced velocity from filtered lifting line theory can be used as a subgrid-scale model to correct the sampled velocity of the actuator line model.¹ At every time step, the induced velocity is computed by using the information from the previous one. To this end, the correction can be derived by taking the difference between the induced velocity from a simulation with ϵ_{LES} and from using ϵ_{opt} . The following algorithm describes how to compute the induced velocity correction at every time step using the information from the previous one for the proposed generalized filtered lifting line theory.

Algorithm 1 Pseudocode for the generalized filtered lifting line theory correction

for every z do

Sample the velocity and add the correction from the previous time $u = u_{LES} + \Delta u$

Using u , compute the angle of attack $\alpha(z)$, $c_l(z)$, and $W(z)$

Compute $G(z) = \frac{1}{2} c_L(z) c(z) W^2(z)$

end for

for every z do

Compute numerically $u_{les}(z) = \frac{1}{2\pi} \int \frac{1}{U_\infty(z')} \frac{G(z')}{\epsilon_{les}(z')^2} \left[e^{-(z'-z)^2/\epsilon_{les}(z')^2} + \frac{\epsilon_{les}(z')^2}{2(z'-z)^2} \left(e^{-(z'-z)^2/\epsilon_{les}(z')^2} - 1 \right) \right] dz'$

Compute numerically $u_{opt}(z) = \frac{1}{2\pi} \int \frac{1}{U_\infty(z')} \frac{G(z')}{\epsilon_{opt}(z')^2} \left[e^{-(z'-z)^2/\epsilon_{opt}(z')^2} + \frac{\epsilon_{opt}(z')^2}{2(z'-z)^2} \left(e^{-(z'-z)^2/\epsilon_{opt}(z')^2} - 1 \right) \right] dz'$

$\Delta u(z) = u_{opt}(z) - u_{les}(z)$

end for
