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Exploiting Power Flow Manifold to Solve AC Optimal Power Flow

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Motivation

Motivation

We want to

1. Reliably solve ACOPF problems for large-scale power systems
2. Deploy these algorithms on accelerators

Both can be challenging due to the numerical linear algebra problem at the core of interior point methods [1, 2, 3].

Optimizing over a Manifold

Smooth Manifolds

We consider a smooth manifold $\mathcal{M}$ to be given by

$$\mathcal{M} = \{x \in \mathbb{R}^n : q(x) = 0\}, \tag{1}$$

where $q : \mathbb{R}^n \to \mathbb{R}^m$ is infinitely differentiable, $m < n$ and Dq_x, the Jacobian of q evaluated at x, is of full-rank for all $x \in \mathbb{R}^n$.

This is called an embedded submanifold of Euclidean space.

We limit ourselves to this case but these concepts can be generalized to a more abstract setting. See [4, 5] for a rigorous mathematical approach or [6] for an optimization oriented discussion.

Tangent Space

Given a point $x \in \mathcal{M}$, consider any smooth curve $\gamma : I \to \mathcal{M}$ where $I \subset \mathbb{R}$ contains zero and $\gamma(0) = x$. The tangent space $T_x\mathcal{M}$ is defined by

$$T_x\mathcal{M} := \{v \in \mathbb{R}^n : v = \gamma'(0)\}. \quad (2)$$

This coincides with the kernel of the Jacobian

$$T_x\mathcal{M} = \{v \in \mathbb{R}^n : Dq_x v = 0\}. \quad (3)$$

The tangent bundle is the disjoint union of all tangent spaces

$$T\mathcal{M} = \{(x, v) : x \in \mathcal{M}, v \in T_x\mathcal{M}\}. \quad (4)$$

Riemannian Manifold

We pair $\mathcal{M}$ with a Riemannian metric

$$g_x : T_x\mathcal{M} \times T_x\mathcal{M} \to \mathbb{R} \tag{5}$$

to get a Riemannian Manifold. The Riemannian metric generalizes inner products to a manifold.

We take g_x to be the standard Euclidean inner product

$$g_x(u, v) = \langle u, v \rangle_x := \sum_{i=1}^{n} u_i v_i \tag{6}$$

where $u, v \in T_x\mathcal{M}$.

Riemannian Gradient

Let $f : \mathcal{M} \to \mathbb{R}$. The Riemannian gradient of f is the unique vector field $\text{grad} f$ on $\mathcal{M}$ such that for all $(x, v) \in T\mathcal{M}$, we have

$$Df(x)[v] = \langle \text{grad} f(x), v \rangle_x \tag{7}$$

where Df is the differential of f. For a manifold given by (1), we have

$$\text{grad} f(x) = P_x(\nabla \hat{f}(x)) \tag{8}$$

where $\hat{f}$ is any smooth extension of f to $\mathbb{R}^n$, ∇ denotes the standard Euclidean gradient and $P_x : \mathbb{R}^n \to T_x\mathcal{M}$ is the orthogonal projection and is given by the matrix

$$P_x = I - Q_x, \quad Q_x = Dq_x(Dq_x Dq_x^T)^{-1} Dq_x^T. \tag{9}$$

Riemannian Optimization

We are interested in solving

$$\min_{x \in \mathcal{M}} f(x) \quad \text{s.t.} \quad h(x) \leq 0. \tag{10}$$

How do we generalize Euclidean algorithms?

We need a few extra concepts.

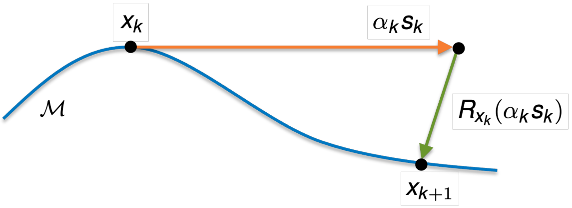
Retraction

A retraction is a smooth map $R : T\mathcal{M} \to \mathcal{M} : (x, v) \to R_x(v)$ such that for each curve $\gamma(t) = R_x(tv)$ we have $\gamma(0) = x$ and $\gamma'(0) = v$.

A retraction is used to ensure iterates of any optimization algorithm are on the manifold,

$$x_{k+1} = R_{x_k}(\alpha_k s_k), \tag{11}$$

where $x_k \in \mathcal{M}$, $s_k \in T_{x_k}\mathcal{M}$ and $\alpha_k \in \mathbb{R}$.



Vector Transport

A vector transport on $\mathcal{M}$ is a smooth, linear map

$$\mathcal{T} : T\mathcal{M} \bigoplus T\mathcal{M} \to T\mathcal{M} : (u, v) \to \mathcal{T}_u(v) \tag{12}$$

such that, for all $x \in \mathcal{M}$ and for all $u, v \in T_x\mathcal{M}$, there exists a retraction R where

$$\mathcal{T}_u(v) \in T_{R_x(u)}\mathcal{M} \quad \text{and} \quad \mathcal{T}_0(v) = v. \tag{13}$$

A vector transport is used to move a vector from one tangent space to another. For example,

$$\text{grad} f(x_{k+1}) - \mathcal{T}_{\alpha_k s_k}(\text{grad} f(x_k)). \tag{14}$$

Application to AC Optimal Power Flow

Power Flow Manifold

We can (briefly) write the ACOPF as

$$\min_{(s_g,u)\in\mathbb{R}^{4n}} f(s_g, u) \tag{15}$$

such that

$$diag(u)\overline{Yu} - s = 0, \tag{16}$$

$$s = s(s_g, s_d), \tag{17}$$

$$h(s_g, u) \leq 0. \tag{18}$$

Equation (16) creates an embedded submanifold of Euclidean space [7].

Computational Setup and Results

Numerical Benchmarks

Used PowerModels.jl [8] to create ACOPF problems. Selected smaller cases from pglib-opf repository [9].

Used Ipopt [10] as a benchmark.

Tested Algorithms

Inequality constraints are handled using

1. Riemannian Augmented Lagrangian (RAL) from [11]
2. Riemannian Exact Penalty (REP) from [11]

Subsolve is handled using

1. Riemannian Gradient Descent (RGD) from [6]
2. Riemannian Conjugate Gradient (RCG) from [12]
3. Riemannian Quasi-Newton (RQN) from [13]

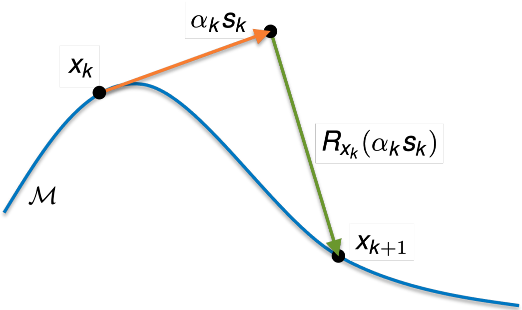
All these methods are implemented in Manopt.jl [14].

Retraction

We use the orthographic retraction as presented in [15]. We perform the iteration

$$y_k^{\ell+1} = y_k^{\ell} - Dq_{x_k}^T (Dq_{x_k} Dq_{x_k}^T)^{-1} q(y_k^{\ell}). \tag{19}$$

This searches for the manifold in a direction perpendicular to $T_x\mathcal{M}$.



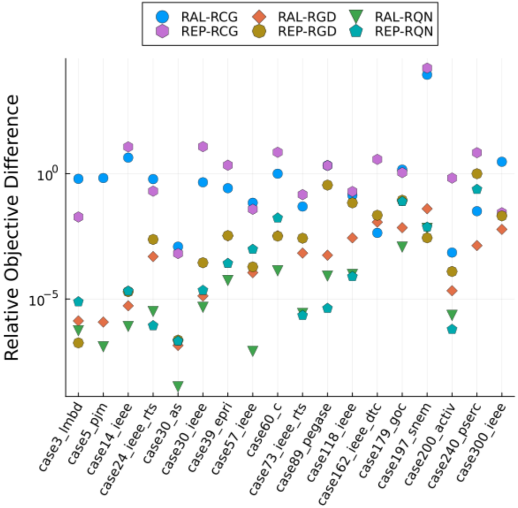
Vector Transport

We use the vector transport $\mathcal{T}$ given by

$$\mathcal{T}_u(v) = P_{R_x(u)} v \tag{20}$$

where $P_y : \mathbb{R}^n \to T_y\mathcal{M}$ is the orthogonal projector (given explicitly by (9)).

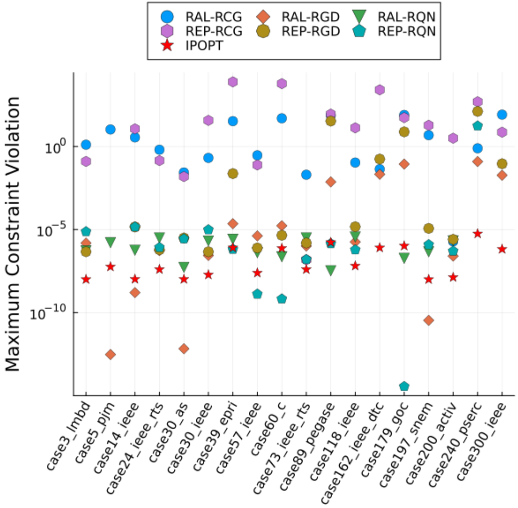
Results: Objective Difference



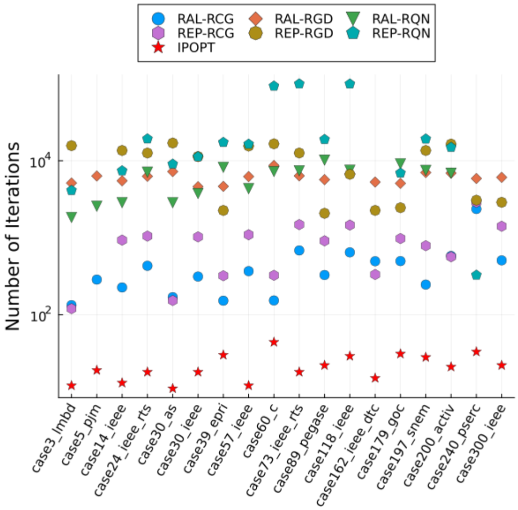
Relative objective difference is given by

$$\frac{f(x_{ro}) - f(x_{ipopt})}{f(x_{ipopt})} .$$

Results: Constraint Violation



Results: Iteration Count



Ongoing and Future Research

Ongoing Research

Computational next steps:

- Test larger systems
- Test other algorithms (e.g., Riemannian Trust Region)
- Implement and test coordinate retraction (the power flow manifold can be realized as a graph).

Theoretical next steps:

- Use Riemannian geometry to develop computable error bounds for linearized power flows (e.g., DCOPF problems)

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Questions?

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