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ANALYTIC EXPRESSIONS FOR MAXIMUM WIND TURBINE AVERAGE POWER IN A RAYLEIGH WIND REGIME[1]

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Abstract

Average or expectation values for annual power of a wind turbine in a Rayleigh wind regime are calculated and plotted as a function of cut-out wind speed. This wind speed is expressed in multiples of the annual average wind speed at the turbine installation site. To provide a common basis for comparison of all real and imagined turbines, the Rayleigh-Betz wind machine is postulated. This machine is an ideal wind machine operating with the ideal Betz power coefficient of 0.593 in a Rayleigh probability wind regime. All other average annual powers are expressed in fractions of that power. Cases considered include: 1) an ideal machine with finite power and finite cutout speed, 2) real machines operating in variable speed mode at their maximum power coefficient, and 3) real machines operating at constant speed.

Introduction

The Carnot Theorem -- the universal yet elegant expression for the maximum efficiency of any heat engine -- has been of great help in guiding the development of modern engines and power plants. According to this theorem, the maximum possible energy conversion efficiency depends only on temperatures, and is independent of other details in any engine. Although such generality is probably impossible for wind energy conversion, it is interesting to see what fundamental limitations can be inferred from a standard wind probability density, an ideal wind machine, and a small number of other assumptions.

The wind machine model that we will examine is governed by the following assumptions. A few additional assumptions will be made later.

- The rotor and power train have no inertia and are therefore at all times in equilibrium with the local wind both in rotational speed and in yaw alignment. There is no friction nor other mechanical loss.
- The local wind speed probability density is given by the Rayleigh density expression. Also assumed is that a single number is sufficient to describe the instantaneous wind at the rotor disk.
- The power coefficient will be $C_p = 16/27$, which is the classical Betz limit. We will later use measured lower but real power coefficients in an expression for the energy capture of real machines operating in a variable-speed mode. Most variable-speed algorithms attempt to hold the machine operating locus at the maximum power point for the existing wind.

The Machine

In order to establish a baseline number with which to compare all other wind machines, both real and mathematical models, let us first consider an ideal machine that has no speed, power, nor endurance limitations. Because the Rayleigh function determines its wind, and because its power coefficient is the Betz limit, we will call this model a Rayleigh-Betz machine.

The expected or mean value of a stochastic variable, y, is:

$$\overline{Y} = \int_0^\infty y \times f(y)\, dy \qquad (1)$$

where f(y) is its probability density. Thus, the average annual power will be:

$$W_t = \int_0^\infty [\tfrac{1}{2} \rho A\, C_p(\omega R/v)\ v^3] \times \eta \times [2v/c^2\ e^{[-(v/c)^2]}]\, dv \qquad (2)$$

The left bracketed term is the instantaneous power available at the hub of a wind machine. The power coefficient, C_p, is a function of the tip-speed ratio ($\lambda = \omega R/v$). When this bracket is multiplied by the drive train efficiency, η, the product becomes the instantaneous power at the wind machine output. The final bracketed term is the Rayleigh probability density. Note that the probability density differential, $dp = [f(v)]dv$, can be thought of as having units of years per year (yr/yr) or hours per hour (h/h). This means that the above integral may either have units of watt-years per year or watt-hours per hour. In either case we multiply the above integral by 8760, the approximate number of hours in a year. This new numerical value is now the total annual energy measured in watt-hours per year. Because of this trivial conversion, we will work only with mean annual power knowing that it is proportional to annual energy capture.

This expression can be improved by inserting the Rayleigh (or Weibull with k = 2) characteristic wind speed, c, in appropriate places. As this is an ideal machine, $\eta = 1$. Also, we replace the power coefficient with the constant Betz value of $C_{pb} = 16/27$:

$$W_t = \tfrac{1}{2}\ \rho A\ c^3\ C_{pb} \int_0^\infty (v/c)^3\ [2v/c\ e^{[-(v/c)^2]}]\, dv/c \qquad (3)$$

We can now normalize the wind speed by defining a dimensionless wind speed, x:

$$x = v/c, \quad \text{so therefore:} \quad dx = dv/c$$

Thus:

$$W_t = \tfrac{1}{2} \rho c^3 A\, C_{pb} \int_0^\infty x^3\ [2x\ e^{[-(x)^2]}]\, dx \qquad (4)$$

We have now succeeded in removing the particular wind machine constants from the integral; thus, the integral's numerical value becomes independent of them. Now we can think of winds at any location being measured in units of their local characteristic wind speed, c.

If we now integrate over all wind speeds, the average annual power in watts from a Rayleigh-Betz machine is:

$$W_t = \tfrac{1}{2} \rho c^3 A\ (16/27)\,(3/4)\sqrt{\pi} \qquad (5)$$

where $(3/4)\sqrt{\pi}$ is the numerical value of the integral.

This expression can be made more useful by replacing the characteristic wind speed, c, by its value in terms of the average annual wind speed, V. At the same time we substitute for disk area, A.

$$\overline{V} = c\sqrt{\pi}/2 \quad or \quad c = 2\overline{V}/\sqrt{\pi} \quad and \quad A = \pi D^2/4 \qquad (6)$$

By combining all constants (including $C_{pb} = 0.593 = 16/27 = 2^4/3^3$), an especially simple expression for annual average Rayleigh-Betz power is obtained:

$$\overline{P} = \rho^1 \left(\frac{2}{3} D\right)^2 \overline{V}^3 \qquad (7)$$

(average annual watts)

Because of average air density to the first power, diameter to the second power, and local average wind to the third power, we will call this result "the one-two-three formula." It is also fortunate that the numerical constant is formed from a 2 and a 3. An example will illustrate this equation: The average annual production of an 18-meter Rayleigh-Betz machine at sea level in a 6 m/s annual mean wind regime would be:

$$W_t = (1.225 \text{ kg/m}^3)\ (2/3 \times 18 \text{ m})^2\ (6 \text{ m/s})^3 \quad (8)$$

$$= 38.1 \text{ kW}$$

Multiplication of this number by 8760 h/yr yields an expected annual energy production of 334,000 kilowatt-hours.

The Capture Coefficient

Because the preceding result depends only on rotor diameter and the wind climate of the wind site, we can as mentioned above use the Rayleigh-Betz machine as a baseline with which any other machine, real or theoretical, can be compared. Let us therefore define the Capture Coefficient (CC) for any wind machine:

$$CC = \frac{\text{(total annual energy from any machine)}}{\text{(total annual energy from a Rayleigh-Betz machine)}} \quad (9)$$

This number should lie between zero and one (100%), and all other things being equal, a higher number is more desirable.

Limited Constant-Power Coefficient Machines

Having found an upper limit for the physically unlikely case of unlimited machine power and speed, we will now employ more realistic operating parameters. Let us consider a perfect machine with finite cutout wind speed. We will rewrite the starting equation for mean power; however, we will now set a finite upper limit or cut-out speed, X_f, on the integral.

The integration eventually results in:

$$W_t = ½\rho c^3 A\, C_{pb} [(3/4)\sqrt{\pi}(erf\ X_f) - X_f(X_f^2 + 3/2)\, e^{-X_f^2}] \quad (10)$$

where erf (X_f) = the error function of X_f. When this quantity is substituted into the expression for the capture coefficient, the result simplifies to:

$$CC(X_f) = erf\ (X_f) - \frac{4}{(3\sqrt{\pi})} X_f\, (X_f^2 + 3/2)\, e^{-X_f^2} \quad (11)$$

Although we could now plot this function, the expression can be made more useful by re-normalizing the wind speed, v. Although normalization by the Weibull characteristic speed c is algebraically convenient, the results are more useful if the local average wind speed is used instead. Let the new independent wind variable be s. In effect we are merely changing the units of the independent variable.

$$\bar{V}/c = \sqrt{\pi}/2 \quad and \quad s = v/\bar{V},$$
$$so \quad x = v/c = v/\bar{V} \times \bar{V}/c = s\ (\bar{V}/c) = [\sqrt{\pi}/2]\, s \quad (12)$$

so that:

$$CC\,(S_f) = erf[\frac{\sqrt{\pi}}{2} S_f] - 2/3\ S_f\, (\pi/4\ S_f^2 + 3/2)\ e^{(-\pi S_f^2)/4} \quad (13)$$

where S_f is the cutout wind speed expressed in units of the local average wind speed.

This equation yields a universal energy production curve upper limit for any machine at any site to within the constraints cited earlier for our model. It is plotted in Figure 1. We can, as a result, see the amount of annual energy we forfeit due to stopping a machine when the wind exceeds a particular cutout value.

Note also that the effect of a finite cut-in wind speed can be accounted for by entering the curve with this cut-in speed, and then subtracting the small percentage found from the capture coefficient given above. Symbolically,

$$\int_0^{s_F} = \int_0^{s_C} + \int_{s_C}^{s_f} \quad so \quad \int_{s_C}^{s_f} = \int_0^{s_F} - \int_0^{s_C} \qquad (14)$$

For example, if operation is inhibited for all wind speeds below the local average wind, only about 10% of the ideal energy is lost.

Machines with Finite Power Ratings

Having taken one step toward realism by accounting for finite cutout speeds, we now define the rated wind speed, V_r, of a machine as that wind speed above which the power output is limited to a constant value. Starting from fundamentals, we can break the energy integral into two parts:

$$\begin{aligned} W_t &= \int_0^{V_f} p(v)\, f(v)\, dv \\ &= \int_0^{V_r} p(v)\, f(v)\, dv + \int_{V_r}^{V_f} P_r f(v)\, dv \end{aligned} \qquad (15)$$

$$\begin{aligned} &= \tfrac{1}{2}\rho c^3 A\, C_{pb} \int_0^{X_r} x^3 \times 2x\, e^{-x^2}\, dx \\ &\quad + P_r \int_{X_r}^{X_f} 2x\, e^{-x^2}\, dx \end{aligned} \qquad (16)$$

where P_r is the power produced by this machine at its rated wind speed. We are assuming that at any speed between rated speed and cutout, the power is rated power.

Substituting directly into the capture coefficient, we have:

$$CC = \frac{\tfrac{1}{2}\rho c^3 A\, C_{pb} \int_0^{X_r} x^3 \times 2x\, e^{-x^2}\, dx}{\tfrac{1}{2}\rho c^3 A\, C_{pb}\, (3/4)\sqrt{\pi}} + \frac{\tfrac{1}{2}\rho c^3 A\, C_{pb}\, X_r^3\, [e^{(-X_r^2)} - e^{(-X_f^2)}]}{\tfrac{1}{2}\rho c^3 A\, C_{pb}\, (3/4)\sqrt{\pi}} \qquad (17)$$

$$= CC(X_r) + \frac{X_r^3}{(3/4)\sqrt{\pi}}\, [e^{(-X_r^2)} - e^{(-X_f^2)}] \qquad (18)$$

Note that the capture coefficient is composed of two parts:

- $CC(X_r)$, which equals the capture coefficient if the machine is stopped at the rated wind speed.
- A second term that depends only on rated and cutout wind speeds, and not on the aerodynamic characteristics of the rotor.

This expression for capture coefficient can now be considered a function of two independent variables: rated wind speed and cutout wind speed. Again rescaling the abscissa to annual average wind speed, we can plot a capture coefficient curve family with curve branches showing the effect of different machine ratings. This was done in the Figure 2. For example, "Limited above 1.4" means for all winds exceeding 1.4 times the annual mean wind yet below cutout, generator power is held constant. Although high winds do have a great deal of power, the curves show that for machines of finite rating, there is negligible additional annual energy in winds beyond approximately three times the average wind in a Rayleigh probability density.

Real Machines with Constant Power Coefficients

Thus far, we have used the ideal power coefficient. If its speed is allowed to vary, a real machine can operate at a constant but non ideal power coefficient, so we modify our previous results to account for this.

Starting with Equation 4 and setting C_p equal to a constant but realistic value, we have at the hub an annual average power of:

$$W_t = \tfrac{1}{2}\rho c^3 A\, C_p \int_0^{x_f} x^3 \times 2x\, e^{-(x^2)}\, dx \qquad (19)$$

Reintegration with the new power coefficient yields:

$$CC = \frac{C_p}{(16/27)} \times \frac{CC(X_r)}{(3/4)\sqrt{\pi}} + \frac{C_p X_r^3}{C_{pb}\,(3/4)\sqrt{\pi}} \left[e^{-X_r^2} + e^{-X_f^2}\right] \qquad (20)$$

In short, we simply degrade the whole ideal CC curve by multiplying it by the actual effective power coefficient normalized to the Rayleigh-Betz maximum value. This curve will be useful for comparison with the constant rotor speed machines treated below.

Constant Rotor Speed Mathematical Model

The most interesting case for comparison is machine operation at a constant rotor speed, such as is found with conventional induction generators. Because this means that in a variable wind the instantaneous tip-speed ratio is continuously changing inversely to the wind, then harvested energy depends on the explicit details of the C_p versus λ (tip-speed-ratio) curve. We therefore need of an analytic expression for power coefficient.

In trying to balance accurate correspondence to the real world with mathematical tractability, we select a double exponential curve to approximate the rotor characteristics:

$$C_p = H \times \left[e^{-\alpha_1 (x - x_0)} - e^{-\alpha_2 (x - x_0)}\right] \qquad (21)$$

Note that there are four constants to approximate a match to given data:

x_0 = cut-in speed locates the wind axis crossing
α_2 = controls curve shape predominantly on the left side
α_1 = controls curve shape predominantly on the right side
H = vertical scale.

The curve plotted in Figure 3 is the above double exponential with the four parameters adjusted for best fit. This exponential curve usually adequately fits most simulated and measured power coefficient versus wind speed curves. The experimental data being fitted in the example shown in Figure 3 are from tests on the NREL variable speed test bed.

Returning again to fundamentals as expressed in previous equations:

$$W_t = \tfrac{1}{2}\rho c^3 A \int_0^{x_f} x^3\, C_p(x/\lambda_c) \times 2x\, e^{-x^2}\, dx \qquad (22)$$

The capture coefficient is:

$$CC = \frac{4}{3\sqrt{\pi}} \int_0^{x_f} x^3\, C_p(x/\lambda_c) \times 2x\, e^{-x^2}\, dx \qquad (23)$$

The fact that the natural independent variable here is normalized wind means that the usual C_p versus λ function must have its independent variable inverted to display wind speed explicitly. Figure 3 shows the power coefficient plotted this way. Note that the λ axis has become proportional to the advance ratio (reciprocal of tip-speed ratio), and the abscissa crossing that marks reversal of power flow is now on the left and becomes, in effect, the minimum useful normalized cut-in wind speed. The right side of this inverted curve represents the stalling and beyond region of machine operation.

Even though the constant-speed machine rarely operates at its best C_p, nevertheless, for a particular location, it is possible to select one constant turbine speed that is preferable to the others. We shall establish that optimal speed for one location as follows. Conceptually operate the constant speed generator at its preferred speed. At the same time, conceptually allow rotation of the wind machine rotor at such a speed that it will be operating at its best tip-speed ratio when the Rayleigh wind is blowing at its most productive speed. For a Rayleigh wind this is at 159% of its annual average. (This most productive wind is at the peak of the curve formed from the product of v^3 and the Rayleigh probability density curve.) Then select a transmission gear ratio compatible with these two speeds. Although this rationale usually provides best energy capture, for some particular shapes of the power coefficient curve that are skewed to the right small (~10%) reductions in this gear ratio can give a few more percent of mean annual power than the procedure just described.

The integration of equation (21) is conventional but onerous. If we accept the previous chain of assumptions about our model, we now have an expression giving capture coefficient as a function of cutout speed.

Figure 4 shows this constant speed function together with the previous real variable speed machine and the Rayleigh-Betz machine. The top curve is the original Rayleigh-Betz ideal curve, and the next lower curve shows the expected variable speed behavior of a real machine. The third curve labeled "Constant Speed" is plotted assuming the preceding variable speed machine is now constrained to a constant speed but with a transmission gear ratio chosen by the procedure described above.

The lowest curve is the ratio of the constant Cp curve to the constant speed curve minus one, thereby showing the theoretical improvement from variable speed operation. For these ideal machines and the Cp versus λ curve used, it shows that for cutout speeds above 3 times the local annual wind speed improvements of 27% to 28.3% in mechanical energy capture for variable speed over constant speed. These curves do not show a rated wind speed that designates the transition to constant power. However, limiting branches can be added here just as in Figure 2.

Conclusions

It is not intended that the material in this report should be used for detailed design of a wind machine nor to tune a given machine to a particular field site. Rather this paper may contribute to technical insight and show how simple physical models can indicate probable *a priori* limits to wind machine performance. Perhaps the numerical results derived here can yield quantitative support for beliefs that some wind professionals may have intuitively held from their practical experience with wind energy. Some possible examples are:

- For the model chosen it is possible to cite a well defined number for the increased mechanical or hub energy capture of variable speed operation of a given turbine over its constant speed performance. This number must be weighed against the losses and costs of any required power electronics. Further, the designer needs to estimate how closely a particular variable speed control algorithm can track the maximum power coefficient in a turbulent wind.

- In any ordinary wind regime the energies harvested from very high winds and winds near cut-in are not worth pursuing.

- A speed range for variable speed wind turbine operation probably need not exceed a two and a half to one range.

The next phase of this work now underway is to include the effect of power electronics on the hub energy in a non-dimensional way.

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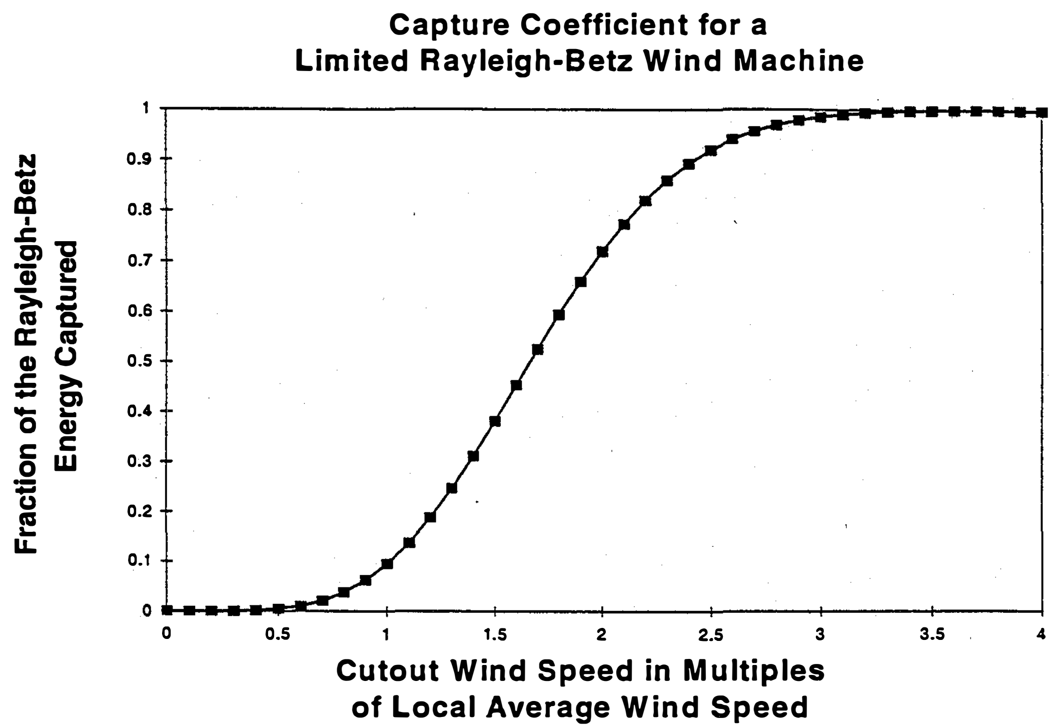

Figure 1

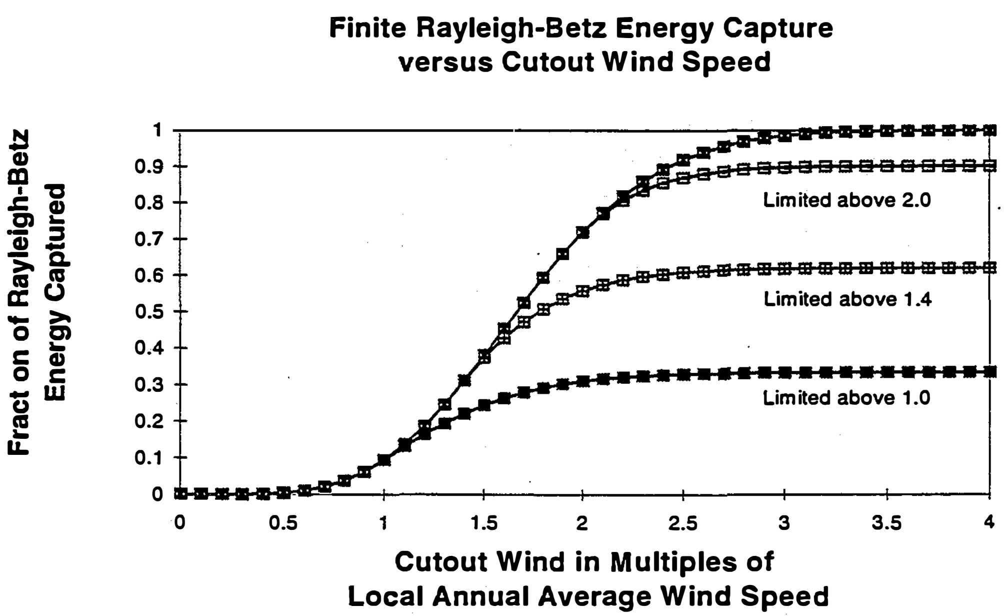

Figure 2

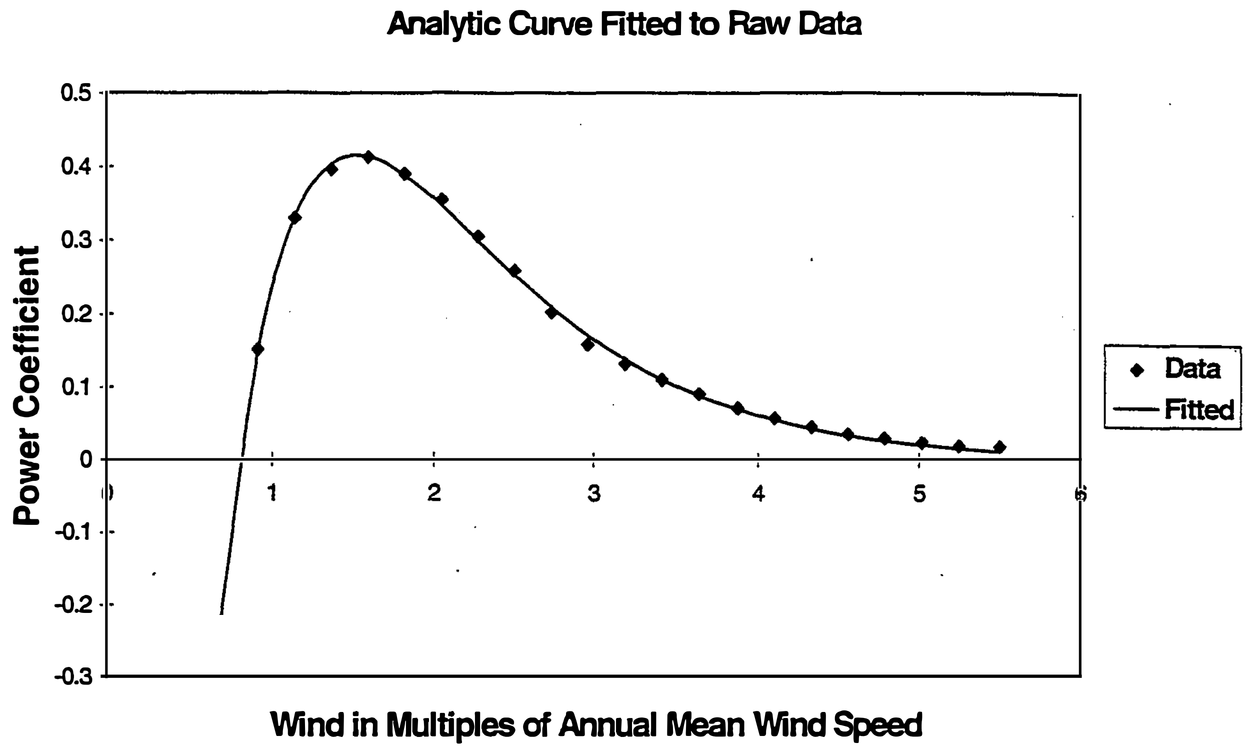

Figure 3

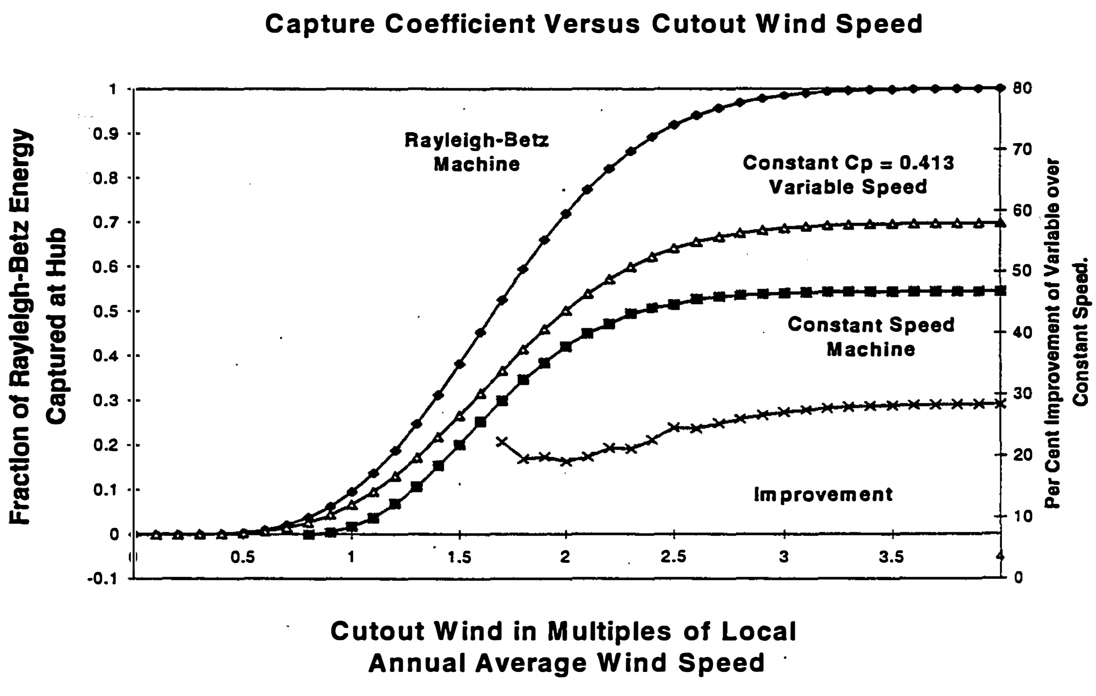

Figure 4